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HEAT FLUX MEASUREMENT FROM THE TEMPERATURE FIELD OF A HEATED SURFACE. 2. INHOMOGENEOUS FLUX

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Based on the previously obtained analytical solutions of the three-dimensional space-time problem of recalculating boundary conditions, algorithms are developed and a computational experiment is carried out to reconstruct the heat flux with arbitrary spatial distribution of the temperature field of the heated surface.

The integral relations from [1] for the heat flux under different boundary conditions:
for a plate whose back side is maintained at a constant temperature:

$$q(t,\rho) = -\frac{\pi k}{a^2 L}\int_0^t d\tau \frac{d}{d\tau}\vartheta_1\left(\frac{1}{2}, \frac{t-\tau}{L^2}a^2\right)\int_{-\infty}^{\infty} d\rho' \frac{T(\rho',\tau)}{(t-\tau)}\exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)}\right), \tag{1}$$

in the semiinfinite body approximation:

$$q(t,\rho) = \frac{1}{2a^2\sqrt{\pi^3}}\int_0^t d\tau \int_{-\infty}^{\infty} d\rho' \frac{T(\rho',\tau)}{\sqrt{(t-\tau)^5}}\exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)}\right) \tag{2}$$

and for a plate whose back side is heat-insulated:

$$q(t,\rho) = -\frac{\pi k}{a^2 L}\int_0^t d\tau \frac{d}{d\tau}\vartheta_3\left(1, \frac{t-\tau}{L^2}a^2\right)\int_{-\infty}^{\infty} d\rho' \frac{T(\rho',\tau)}{(t-\tau)}\exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)}\right), \tag{3}$$

have nonintegrable power singularities in the time coordinate. These singular integrals are meaningful when the characteristic "Abelian" kernel $(\sqrt{t-\tau})^{-n}$ is understood as generalized functions [2]. In order to construct such generalized functions and to introduce the corresponding regularization, we shall analyze the integrands in (1)-(3).

In the first of these relations we single out the kernel

$$\psi(r) = \frac{\exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)}\right)}{\sqrt{(t-\tau)^5}}, \tag{4}$$

which at $\rho = \rho'$ has a power singularity of the type $(t-\tau)^{5/2}$ leading to integral divergence at the specific point $t = \tau$. Replacing the singular function $(t-\tau)^{5/2}$ by the generalized function corresponding to it:

$$(t-\tau)_+^{-5/2} = \begin{cases} 0 & \text{at} \quad \tau \ge t, \\ (t-\tau)^{-5/2} & \text{at} \quad \tau < t, \end{cases}$$

and using the regularization rule [2] for the case where the exponent in (4) is in a band of the poles $n = 2$ and $n = 3$, we obtain in the semiinfinite body approximation that

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$$q(t,\rho) = -\frac{k}{8a^3\sqrt{\pi^3}} \int_0^t d\tau \left[\int_{-\infty}^{\infty} d\rho' \, T(\rho', \tau) \exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)} \right) - \right.$$

$$\left. - 4a^2 \left\{ T(\rho, t)(t-\tau) + \frac{\partial T(\rho, t)}{\partial t}(t-\tau)^2 \right\} \right] (t-\tau)^{-5/2} + \frac{k}{\sqrt{\pi}\, a} \left\{ \frac{T(\rho, t)}{\sqrt{t}} - \frac{\partial T(\rho, t)}{\partial t} \sqrt{t} \right\}. \quad (5)$$

To construct regularizing functionals for a plate of finite thickness, we transform the functions under the integrals, namely, $\vartheta_1(1/2, (t-\tau)a^2/L^2)$ and $\vartheta_3(1, (t-\tau)a^2/L^2)$:

$$\vartheta_1\left(\frac{1}{2}, \frac{t-\tau}{L^2}a^2\right) = \sum_{-\infty}^{\infty} \exp\left[-\frac{a^2\pi^2(t-\tau)}{L^2} \left(\frac{2n+1}{2} \right) \right] =$$

$$= \frac{L}{a\sqrt{t}\sqrt{\pi}} \left[1 + 2\sum_{-\infty}^{\infty} (-1)^n \exp\left(-\frac{n^2L^2}{ta^2} \right) \right], \quad (6)$$

$$\vartheta_3\left(1, \frac{t-\tau}{L^2}a^2\right) = \sum_{-\infty}^{\infty} \exp\left[-\frac{a^2\pi^2(t-\tau)n^2}{L^2} \right] = \frac{L}{a\sqrt{t}\sqrt{\pi}} \sum_{-\infty}^{\infty} \exp\left(-\frac{n^2L^2}{ta^2} \right). \quad (7)$$

Functions (6) and (7) are transformed by the Poisson summation formula [3]. Using (6), (7), we arrive at

$$\lim_{\tau\to t} \frac{\partial\vartheta_3\left(1, \frac{t-\tau}{L^2}a^2\right)/\partial t}{t-\tau} \exp\left\{ -\frac{(\rho-\rho')^2}{4a^2(t-\tau)} \right\} = \lim_{\tau\to t} \psi(\tau)$$

$$\lim_{\tau\to t} \frac{\partial\vartheta_1\left(\frac{1}{2}, \frac{t-\tau}{L^2}a^2\right)/\partial t}{t-\tau} \exp\left\{ -\frac{(\rho-\rho')^2}{4a^2(t-\tau)} \right\} = \lim_{\tau\to t} \psi(\tau),$$

which proves that the kernels of (1)-(3) have the same singularity $(t-\tau)^{-5/2}$ at the point $t = \tau$.

Following the regularization rule [2] for the integrals with power singularities, we obtain the following calculation formulas for the plate whose back side is maintained at a constant temperature

$$q(t,\rho) = -\frac{k}{4a^2\pi L} \int_0^t d\tau \left[(t-\tau)^{3/2} \frac{d}{d\tau} \vartheta_1\left(\frac{1}{2}, \frac{t-\tau}{L^2}a^2 \right) \int_{-\infty}^{\infty} d\rho' \, T(\rho', \tau) \times \right.$$

$$\left. \times \exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)} \right) - L2a\sqrt{\pi} \left\{ T(\rho, t)(t-\tau) + \frac{\partial T(\rho, t)}{\partial t}(t-\tau)^2 \right\} \right] (t-\tau)^{-5/2} +$$

$$+ \frac{k}{a\sqrt{\pi}} \left\{ \frac{T(\rho, t)}{\sqrt{t}} - \frac{\partial T(\rho, t)}{\partial t} \sqrt{t} \right\} \quad (8)$$

or heat-insulated

$$q(t,\rho) = -\frac{k}{4a^2\pi L} \int_0^t d\tau \left[(t-\tau)^{3/2} \frac{d}{d\tau} \vartheta_3\left(1, \frac{t-\tau}{L^2}a^2 \right) \int_{-\infty}^{\infty} d\rho' \, T(\rho', \tau) \times \right.$$

$$\left. \times \exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)} \right) - L2a\sqrt{\pi} \left\{ T(\rho, t)(t-\tau) + \frac{\partial T(\rho, t)}{\partial t}(t-\tau)^2 \right\} \right] (t-\tau)^{-5/2} +$$

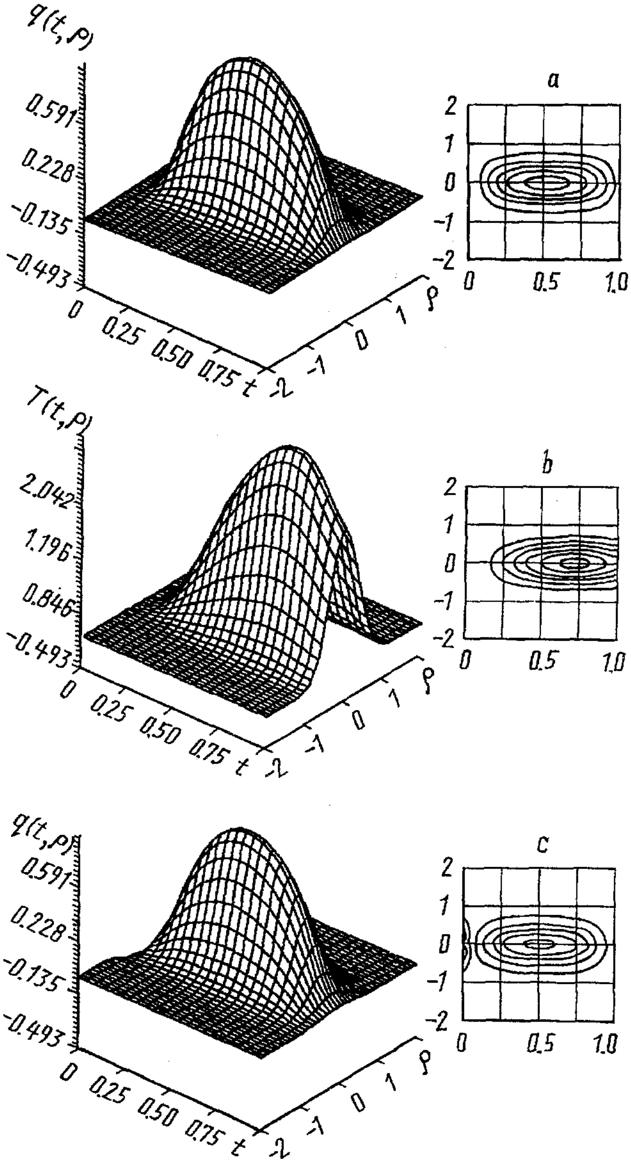

Fig. 1. Heat flux (W/cm^2) reconstruction for the case of the back side of the target maintained at a constant temperature (^{o}C): a) initial model distribution; b) temperature field over the target surface; c) reconstructed heat flux. Normalized values of the variables lie off the coordinates.

$$+\frac{k}{a\sqrt{\pi}}\left\{\frac{T(\rho\,,\,t)}{\sqrt{t}}-\frac{\partial T(\rho\,,\,t)}{\partial t}\sqrt{t}\right\}. \tag{9}$$

The efficiency of heat flux reconstruction by formulas (5), (8), (9) has been checked in a computational experiment. Thermophysical parameters of the experiment are reported in [1]. As models of a heat flux, functions with radial spatial dependence were chosen. This allowed us to calculate only single, instead of double, integrals, in the relations (5), (8), (9) and to reduce the time and memory required for the computation. For the regime of the semiinfinite body we chose the following function

$$q(t\,,\,\rho)=\exp\left\{-a(x^2+y^2)\right\}\Theta(\tau)f_1(\tau)\,, \tag{10}$$

where

$$\Theta(\tau)=\left\{\begin{matrix}1\,, & \tau\geq 0\,, & 0\,,\\ \tau<0\,, & \tau\geq 1\,, & \end{matrix}\right. \tag{11}$$

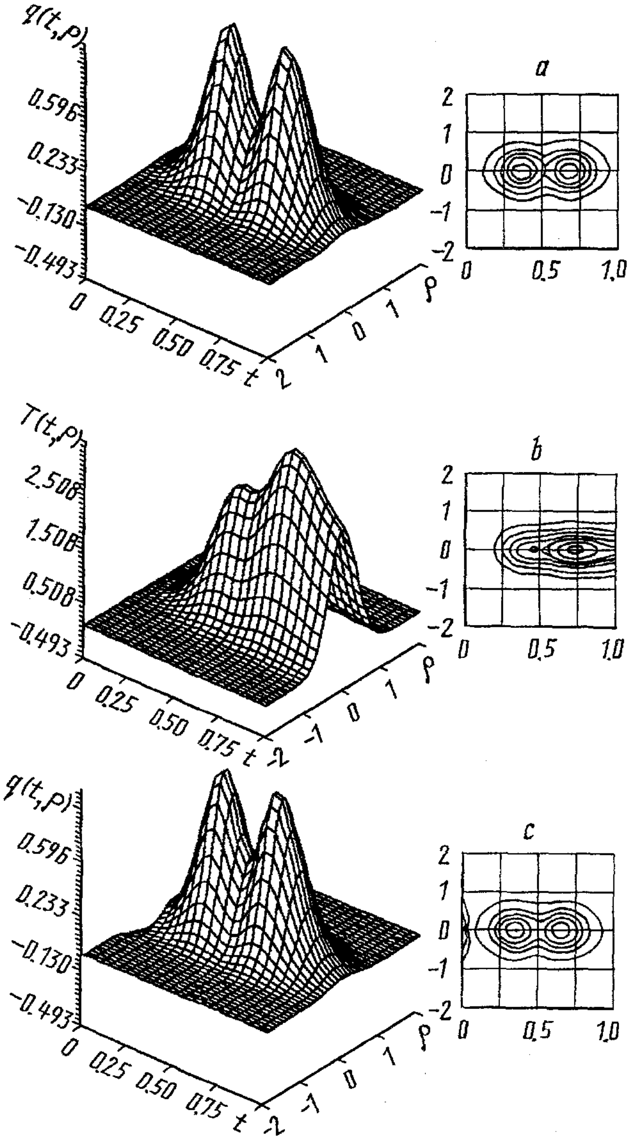

Fig. 2. Heat flux reconstruction for the heat-insulated target: a) initial model distribution; b) temperature field over the target surface; c) reconstructed heat flux.

$$f_1(\tau) = I_0 \left\{ 17\tau^4 - 32\tau^3 + 14\tau^2 + 1 \right\}, \tag{12}$$

$$I_0 = 1 \ \ \mathrm{W/cm^2}, \quad \tau = t/t_0, \quad t_0 = 1 \ \ \mathrm{sec}.$$

The assumption that the heat flux is independent of the radius has also allowed us to simplify calculation of the space-time dependence of the temperature which for $q(t, \rho)$ prescribed by (10)-(12) has the form

$$T(\rho, t) = \frac{a}{k\sqrt{\pi}} \int_0^t d\tau \frac{\exp\left\{ -\rho^2 \dfrac{\alpha}{1 + \alpha 4a^2 (t-\tau)} \right\}}{\sqrt{t-\tau}\ (1 + \alpha 4a^2 (t-\tau))} f_1(\tau). \tag{13}$$

For a cooled plate we chose the spatial time-dependent model of the heat flux (10) but with the time dependence differing from (12)

$$f_2(\tau) = I_0 \exp\left\{ -\frac{(\tau - 0.5)^2}{(0.5)^2 - (\tau - 0.5)^2} \right\}, \tag{14}$$

$$T(\rho, t) = \frac{a^2\pi}{Lk}\int_0^t d\tau \frac{\vartheta_1\left(\frac{1}{2}, \frac{t-\tau}{L^2}a^2\right)\exp\left\{-\rho^2\frac{\alpha}{1+\alpha 4a^2(t-\tau)}\right\}}{(1+\alpha 4a^2(t-\tau))} f_2(\tau), \quad (15)$$

Finally, for the heat-insulated target we write the following flux model and the formula for temperature calculation:

$$q(t, \rho) = \exp(-\alpha\rho^2)\,\Theta(\tau) f_3(\tau),$$

$$f_3(\tau) = 2.6 I_0 [\exp((\tau - 0.5)^2 \alpha_2) - 0.8 \exp((\tau - 0.5)^2 \alpha_3)], \quad (16)$$

$$T(\rho, t) = \frac{a^2\pi}{Lk}\int_0^t d\tau \frac{\vartheta_3\left(1, \frac{t-\tau}{L^2}a^2\right)\exp\left\{-\rho^2\frac{\alpha}{1+\alpha 4a^2(t-\tau)}\right\}}{(1+\alpha 4a^2(t-\tau))} f_3(\tau). \quad (17)$$

In all three models $\alpha = 4\ln(10)$, $\alpha_2 = 20$, $\alpha_3 = 60$ were adopted. Integrals (5), (8), (9), (13), (15), (17) were calculated by the Gaussian method which accounted for the behavior of the integrand at the boundary points. Temperature values were assumed to be recorded, as in [1], with the time resolution $\Delta t = 1/24$ sec. Integration limits over the space variable $\rho\,[0, 2]$ were chosen in such a way as to prevent the temperature at the end of the interval $\rho = 2$ from exceeding 10^{-8} of its maximum value at $\rho = 0$. The number of countings with respect to ρ was equal to 23. Infinite summation was limited by the prescribed error, whichdid not exceed 1% in the all cases. Calculations were made on an IBM PC/AT with an Intel 80386/80387 processor, at a step frequency of 25 mHz, with counting time of, approximately, 35 sec in reconstruction of the instantaneous distribution of the heat flux.

Calculation results for different thermophysical conditions are shown in Figs. 1 and 2. These results show the satisfactory quality of heat flux reconstruction. As is seen, the model and the reconstructed heat fluxes differ only in the presence of artefacts (Figs. 1, 2c) at the initial moment of time. A more sophisticated analysis shows that the reconstruction error does not exceed, on the average, 10%. It is worth noting that the influence of measurement errors on reconstruction quality has not been taken into consideration in this numerical experiment. As shown in [1], this influence, if substantial, may be compensated by smoothing, correspondingly, the experimental data accounting for the level of the measurement error.

Thus, we have implemented a numerical-analytical approach to the problem of reconstruction of the heat flux distribution by means of the temperature distribution on the surface of the heated target. The problem of regularization of the multidimensional-inverse problem of recalculation of boundary conditions has been solved. The algorithms developed can be employed to create a software for the corresponding measuring systems.

NOTATION

$T(t, \pi)$, temperature over the plate surface; $q(t, \rho)$, heat flux to the plate surface; $\rho = \{x, y\}$, transverse coordinate; t, time; k, thermal conductivity; a^2, thermal diffusivity; L, plate thickness; $\vartheta_1(\eta, \xi) = 2\sum_{k\in z}(-1)\exp(-\pi^2(k-1/2)^2 \sin[\pi\eta(2k+1)]$, L, Jacobi theta-function [4]; $\mathrm{Fo} = \alpha^2 t^2/L^2$, Fourier parameter; $\vartheta_3(\eta, \xi) = 2\sum_{k\in z}\exp((-\pi k^2\xi) + i2\pi k\eta)$, Jacobi theta-function [4]; $\Delta_\perp = \partial/\partial x^2 + \partial/\partial y^2$, transversal Laplacian.

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